PERTH MODERN SCHOOL
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Independent Public School

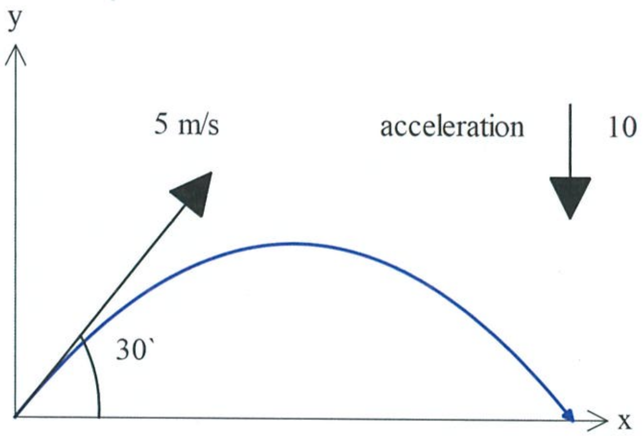
Year 12 Specialist
TEST 4
27 July 2018
TIME: 50 minutes working
NO Classpads NOR calculators allowed!
50 Marks 7 Questions

Name: SOLUTIONS

Teacher: ____________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (2, 1, 3, 3 & 2 = 10 marks)



A particle is projected with an initial speed of $5m/s$ at 30° to the horizontal. The particle experiences a constant downward acceleration of $10m/s^2$.
Determine

i) the initial velocity of the particle in $i - j$ form.

$$\begin{pmatrix} 5\cos 30^\circ \\ 5\sin 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{5\sqrt{3}}{2} \\ \frac{5}{2} \end{pmatrix} \text{ m/s}$$

✓ i component
✓ j component

ii) the position vector, r, t second after projection.

✓ integrate to find $\dot{r}$
✓ uses initial velocity as constant
✓ integrates to find $\underset{\sim}{r}$

$$\ddot{\underset{\sim}{r}} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}} = \begin{pmatrix} 0 \\ -10t \end{pmatrix} + \begin{pmatrix} \frac{5\sqrt{3}}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5\sqrt{3}}{2} \\ \frac{5}{2} - 10t \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} \frac{5\sqrt{3}}{2}t \\ \frac{5}{2}t - 5t^2 \end{pmatrix}$$

iii) the cartesian equation of the path.

$x = \frac{5\sqrt{3}}{2}t$ ✓ $y = \frac{5}{2}t - 5t^2$ ✓

$t = \frac{2x}{5\sqrt{3}}$ $y = \frac{x}{\sqrt{3}} - 5\left(\frac{4x^2}{5^2(3)}\right)$ ✓ (No need to simplify)

iv) the range, that is the distance along the x axis when the particle lands.

$y = \frac{x}{\sqrt{3}} - \frac{4x^2}{15}$

$0 = \frac{x}{\sqrt{3}} - \frac{4x^2}{15}$ ✓

$\frac{4x^2}{15} = \frac{x}{\sqrt{3}}$

(No need to simplify)

$x = \frac{15}{4\sqrt{3}}$ m ✓ (or $\frac{5}{4}\sqrt{3}$)

Q2 (2, 3, 3 & 3 = 11 marks)

An object moves such that its position vector, r metres, at time t seconds is given by

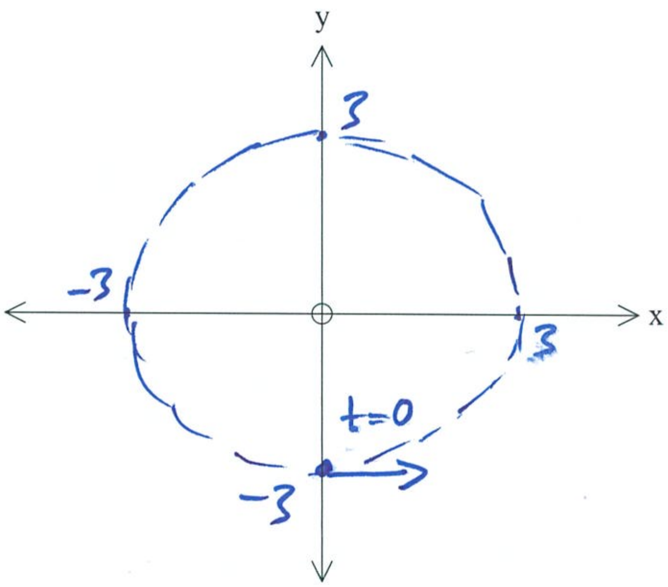
$$r = \begin{pmatrix} 3\sin(4\pi t) \\ -3\cos(4\pi t) \end{pmatrix}$$

i) Determine the cartesian equation of the path of the object and the period of the motion.

$$x^2 + y^2 = 9$$

✓ identifies circle
✓ correct equation

ii) Sketch the cartesian path giving the initial position and direction.



✓ sketches circle of radius 3 unit
✓ initial position at (0,-3)
✓ initial velocity to the right shown

iii) Show that the velocity is always perpendicular to the position vector.

$$\dot{\underset{\sim}{r}} = \begin{pmatrix} \pi 12\cos 4\pi t \\ 12\pi \sin 4\pi t \end{pmatrix} ✓ \quad \underset{\sim}{r} = \begin{pmatrix} 3\sin 4\pi t \\ -3\cos 4\pi t \end{pmatrix}$$

$$\dot{\underset{\sim}{r}} \cdot \underset{\sim}{r} = 36\pi \cos 4\pi t \sin 4\pi t - 36\pi \cos 4\pi t \sin 4\pi t = 0 \; ✓$$

✓

iv) Show that the acceleration is directly proportional to the position vector, stating the constant of proportionality (i.e $\ddot{r} = -k\,r$ where k is a constant)

$$\ddot{\underset{\sim}{r}} = \begin{pmatrix} -48\pi^2 \sin 4\pi t \\ 48\pi^2 \cos 4\pi t \end{pmatrix} ✓ = -16\pi^2 \begin{pmatrix} 3\sin 4\pi t \\ -3\cos 4\pi t \end{pmatrix} ✓ = -16\pi^2 \underset{\sim}{r}$$

$K = 16\pi^2$ ✓ (accept $-16\pi^2$)

Q3 (3 & 3 = 6 marks)

Consider the curve $x^2 = \cos(y)$. In terms of x & y determine an expression for

i) $\dfrac{dy}{dx}$

$2x = -\sin y \, y'$ ✓ ✓ $\quad y' = -\dfrac{2x}{\sin y}$ ✓

ii) $\dfrac{d^2y}{dx^2}$

$2 = -\sin y \, y'' + y'(-\cos y \, y')$ ✓

$\sin y \, y'' = -\cos y \, (y')^2 - 2$

$y'' = \dfrac{-\cos y \, (y')^2}{\sin y} - \dfrac{2}{\sin y}$ ✓

$y'' = \dfrac{-\cos y \, 4x^2}{\sin^3 y} - \dfrac{2}{\sin y}$ ✓

Q4 (3 & 3 = 6 marks)

Show every step in evaluating the following integrals.

i) $\int (5x+1)(3x-2)^7 \, dx$ with substitution $u = 3x-2$ $\quad x = \dfrac{u+2}{3}$

$\int \left\{ \dfrac{5(u+2)}{3} + \dfrac{3}{3} \right\} u^7 \dfrac{1}{3} \, du$ ✓ $= \dfrac{1}{9} \int (5u+13)u^7 \, du = \dfrac{1}{9} \int 5u^8 + 13u^7 \, du$ ✓

$= \dfrac{1}{9} \left[\dfrac{5u^9}{9} + \dfrac{13u^8}{8} \right] + C = \dfrac{5u^9}{81} + \dfrac{13u^8}{72} + C$

ii) $\int \sin^3(2x)\cos^4(2x) \, dx$

$= \dfrac{5}{81}(3x-2)^9 + \dfrac{13}{72}(3x-2)^8 + C$ ✓

No need to factorise.

$\int (1-\cos^2 2x) \sin 2x \cos^4 2x \, dx$ ✓

$= \int \sin 2x \cos^4 2x - \sin 2x \cos^6 2x \, dx$ ✓

$= A\cos^5 2x + B\cos^7 2x + C = -\dfrac{1}{10}\cos^5 2x + \dfrac{1}{14}\cos^7 2x + C$ ✓

Diff $\quad 5A\cos^4 2x(-2\sin 2x) + 7B\cos^6 2x(-2\sin 2x)$

$1 = -10A \qquad -1 = -14B$

$A = -\dfrac{1}{10} \qquad B = \dfrac{1}{14}$

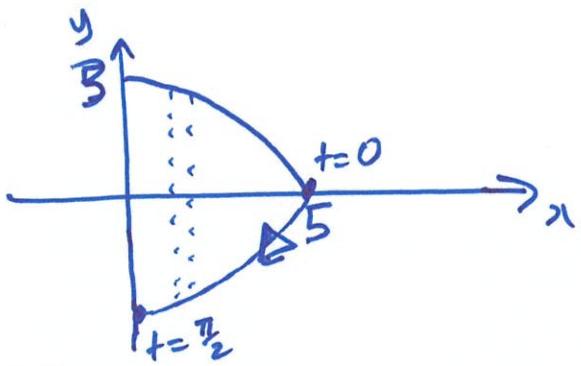
Q5 (4 & 3 = 7 marks)
Consider the curve described parametrically by

$$x = 5\cos t \quad y = -3\sin t \quad \text{from } t = 0 \text{ to } t = \frac{\pi}{2}$$

If this curve is revolved around the x axis a three dimensional shape is formed.

i) Show that the volume of this three dimensional shape is $\int_0^{\frac{\pi}{2}} 45\pi \sin^3 t \, dt$

(Hint- consider direction of integration)



$$V = \int_0^5 \pi y^2 \, dx$$

$$= \int_{\frac{\pi}{2}}^{0} \pi (-3\sin t)^2 \frac{dx}{dt} \, dt$$

$$= \int_0^{\frac{\pi}{2}} 9\pi \sin^2 t (5\sin t) \, dt$$

$$= \int_0^{\frac{\pi}{2}} 45\pi \sin^3 t \, dt$$

✓ $\int \pi y^2 dx$
✓ $\int \pi y^2 \frac{dx}{dt} dt$
✓ correct limits in correct order $\int_0^{\frac{\pi}{2}}$
✓ final expression.

ii) Evaluate this integral to determine the exact volume.

$$\int_0^{\frac{\pi}{2}} 45\pi (1 - \cos^2 t) \sin t \, dt \;✓\; = \int_0^{\frac{\pi}{2}} 45\pi \sin t - 45\pi \cos^2 t \sin t \, dt$$

$$= 45\pi \left[A\cos t + B\cos^3 t \right]_0^{\frac{\pi}{2}}$$

$$\left\{ \text{diff} \quad -A\sin t \quad -3B\cos^2 t(\sin t) \right\}$$

$-A = 1 \quad \therefore A = -1$

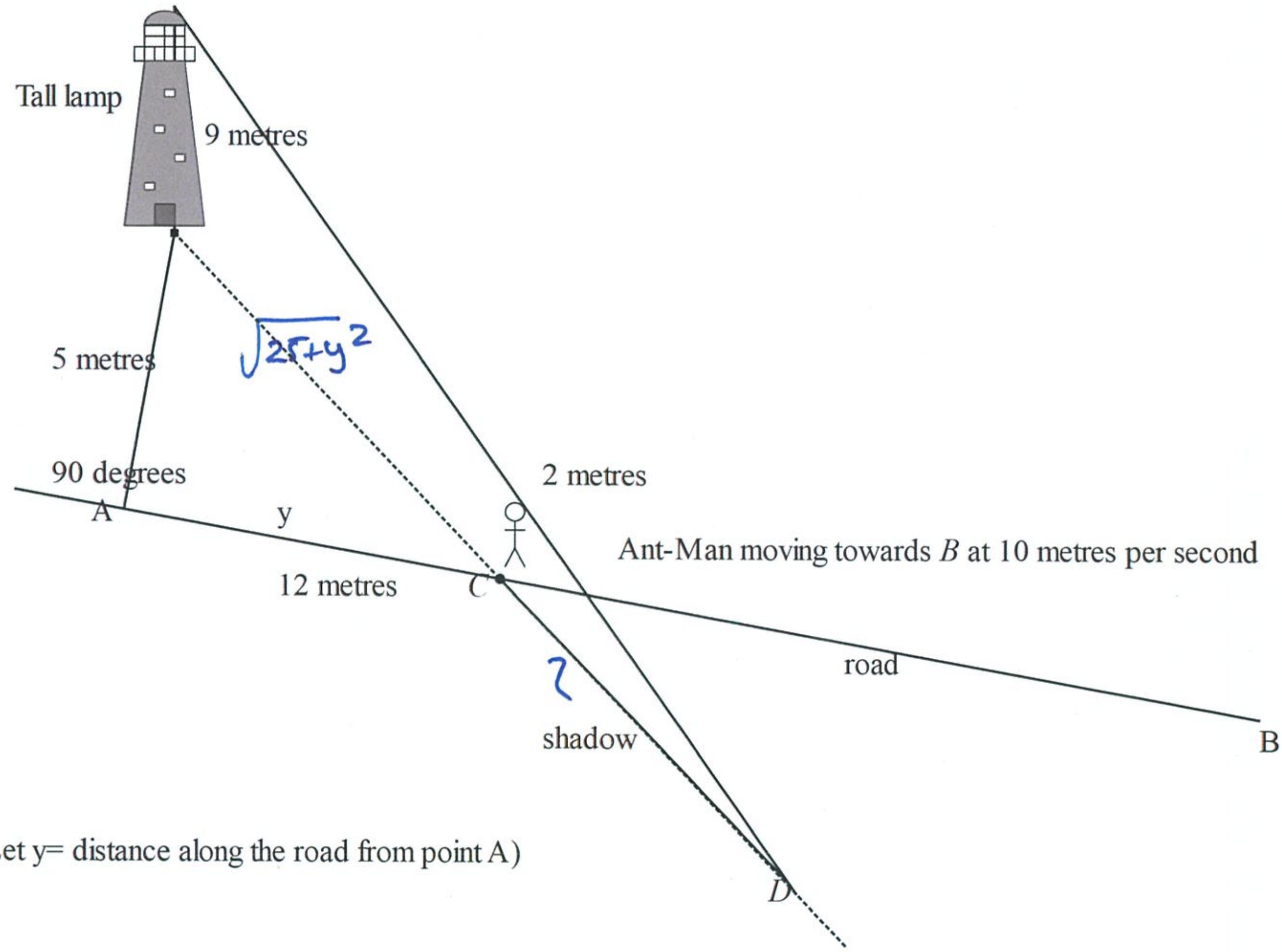
$-3B = -1 \quad B = \frac{1}{3}$

$$= 45\pi \left[-\cos t + \frac{1}{3}\cos^3 t \right]_0^{\frac{\pi}{2}} \;✓$$

$$= 45\pi \left[0 - \left(-1 + \frac{1}{3}\right) \right] = 30\pi \;✓$$

Q6 (6 marks)

Consider the Ant-Man walking along a road AB towards point B at an incredible constant speed of $10m/s$. The height of the Ant-Man is 2 metres. Let point A be the closest point of the base of the Tall lamp from the road, i.e 5 metres and the height of the Tall lamp being 9 metres.



(Hint- Let y= distance along the road from point A)

Determine at the point where the Ant-Man is **12 metres along the road** from point A, the time rate of change of the length of the shadow CD.

$$\frac{l}{l+\sqrt{25+y^2}} = \frac{2}{9}$$

$$9l = 2l + 2\sqrt{25+y^2}$$

$$7l = 2\sqrt{25+y^2}$$

$$7\dot{l} = (25+y^2)^{-\frac{1}{2}}\, 2y\dot{y}$$

$$7\dot{l} = \frac{1}{13}\, 2(12)(10)$$

$$\dot{l} = \frac{240}{13(7)} \text{ m/s} \left(\text{or } \frac{240}{91}\right)$$

✓ uses similar triangles

✓ uses $\sqrt{25+y^2}$

✓ determines expression linking length to y

✓ uses implicit diff to link $\dot{l}$ with $\dot{y}$

✓ subs correct values for y & $\dot{y}$

✓ determines $\dot{l}$

Q7 (4 marks)

By using an appropriate substitution **and** integration, show that

$$\int \frac{\sin x}{1-\cos^2 x}\,dx = \frac{1}{2}\ln\left(\frac{\cos x - 1}{\cos x + 1}\right) + c$$

Let $u = \cos x$

$$\int \frac{\sin x}{1-u^2}\,\frac{1}{(-\sin x)}\,dx$$

$$= \int \frac{1}{u^2-1}\,du$$

$$= \int \left\{ \frac{A}{u-1} + \frac{B}{u+1} \right\} du$$

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1)$$

$u = 1$

$1 = 2A \quad A = \frac{1}{2}$

$u = -1$

$1 = -2B \quad B = -\frac{1}{2}$

$$= \frac{1}{2}\int \frac{1}{u-1} - \frac{1}{u+1}\,du$$

$$= \frac{1}{2}\left[\ln(u-1) - \ln(u+1)\right] = \frac{1}{2}\ln\frac{u-1}{u+1}$$

$$= \frac{1}{2}\ln\frac{\cos x - 1}{\cos x + 1}$$

$$= RHS$$

✓ uses $u = \cos x$

✓ obtains $\frac{1}{u^2-1}$

✓ uses partial fractions AND obtains correct coefficients

✓ uses natural logs + appropriate laws to obtain final answer